Chapter 13
Bells Inequalities in Quantum Optics

Abstract The early days of quantum mechanics were characterised by debates over the applicability of established classical concepts, such as position and momentum, to the new formulation of mechanics. The issues became quite distinct in the protracted exchange between A. Einstein and N. Bohr, culminating in the paper of *Einstein, Podolsky* and *Rosen* (EPR) in 1935 [1]. *Bohr*, in his response to this paper, [2] expanded upon his concept of complementarity and showed that the EPR argument did not establish the incompleteness of quantum mechanics, as EPR had claimed, but rather highlighted the inapplicability of classical modes of description in the quantum domain. A. Einstein, however, did not accept this position and the two sides of the debate remained unreconciled, while most physicists generally believed that N. Bohr's argument carried the day.

Thus the matter rested until 1964 when J.S. Bell opened up the possibility of directly testing the consequences of the EPR premises. We will discuss the EPR argument and the analysis of Bell in the context of correlated photon states.

13.1 The Einstein–Podolsky–Rosen (EPR) Argument

The essential step in the EPR argument is to introduce correlated pure states of two particles (or photons) of the form

$$|\Psi\rangle = \sum_n |a_n\rangle_1 \otimes |b_n\rangle_2 \,, \tag{13.1}$$

where $\{|a_n\rangle_1\}$ and $\{|b_n\rangle_2\}$ are ortho-normal eigenstates for some operators $\hat{A}_1$ and $\hat{B}_2$ of particles 1 and 2, respectively. The correlations between the particles persist even if in the course of the experiment the particles become spatially separated after the interaction responsible for the correlated state.

Now suppose one were to measure the operator $\hat{A}_1$ on particle one long after the interaction between the particles has ended, and the two particles are far apart. If the result is some eigenvalue a_n, particle 1 must thence-forth be considered to be in the state $|a_n\rangle_1$, while particle 2 must be in the state $|b_n\rangle_2$. As the state of particle 2 is now an eigenstate of $\hat{B}_2$ we can predict with probability one that the physical quantity represented by $\hat{B}_2$ if measured will give the result b_n. Thus we can predict the value of this physical quantity for particle 2 without in any way interacting with it.

Suppose, however, that instead of measuring $\hat{A}_1$ on particle 1 we measured some other quantity, $\hat{C}_1$, with eigenstates $|c_n\rangle_1$. We then rewrite the state in (13.1) as

$$|\Psi\rangle = \sum_n |c_n\rangle_1 \otimes |d_n\rangle_2 \,, \tag{13.2}$$

where $|d_n\rangle_2$ is an eigenstate of some other operator $\hat{D}_2$ for particle 2. If the result c_n is obtained for the measurement on particle 1, particle 2 must be in the state $|d_n\rangle_2$ for which a measurement of $\hat{D}_2$ must give the result d_n. Thus depending on what we choose to measure on particle 1 the state of particle 2 after the measurement, can be an eigenstate of two quite different operators. This is another example of the measurement ambiguity discussed in the previous chapter. However, the EPR argument now raises one very important question. Is it possible that the two operators on particle 2, $\hat{B}_2$ and $\hat{D}_2$, do not commute? If this were the case the EPR argument establishes that, depending on what is measured on particle 1, we can predict with certainty the values of physical quantities, represented by noncommuting operators without in anyway interacting with this particle. By explicit construction *Einstein, Podolsky* and *Rosen* showed that this is indeed possible.

EPR claimed that "if without in anyway disturbing a system, we can predict with certainty (i.e., with probability equal to unity), the value of a physical quantity, then there exists an element of physical reality corresponding to that quantity".

Assuming that the wave function does contain a complete description of the two-particle system it would seem that the argument of EPR establishes that it is possible to assign two different states ($|b_n\rangle_2$ and $|d_n\rangle_2$) to the same reality. However, in the language of EPR, two physical quantities represented by operators which do not commute cannot have simultaneous reality. The conclusion of EPR was that the quantum mechanical description of physical reality given by the wave function is not complete.

13.2 Bell Inequalities and the Aspect Experiment

Were one to adopt the conclusion of EPR it would seem necessary to search for a more complete physical theory than quantum mechanics. To obtain such a theory, quantum mechanics should be supplemented by additional, perhaps inaccessible, variables. As *Bell* [3] showed, attempting to complete the theory in this way and maintain the locality condition (that measurements on particle 1 carried out when the particles are spatially separated should have no effect on particle 2) leads to statistical predictions which differ from those of standard quantum theory.

To elucidate *Bell*'s argument we consider a system in which correlated photon polarisation states are produced. Such a system is the $(J=0) \to (J=1) \to (J=0)$

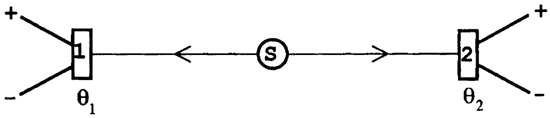


Fig. 13.1 Schematic representation of the experiment of *Aspect* et al. [4] to test quantum mechanics against the Bell inequality. S is a source of two polarised photons. 1 and 2 label *polarisation analysers*, with orthogonal output channels labelled $+$ and $-$. The polarisation analysers are set at angles θ_1, θ_2

cascade two-photon transition in calcium-40 (Fig. 13.1). The two photons are emitted in opposite directions (by conservation of momentum) with correlated polarisation states. Each photon passes through separate polarisation analysers, emerging in either the horizontal $(+)$ channel, or the vertical channel $(-)$ of each analyser. Initially let us assume that the horizontal polarisation is chosen to be orthogonal to the plane of the experiment and that both analysers are so aligned. However, we are free to rotate the polarisers in the plane orthogonal to the propagation direction of the photons. We follow the treatment of *Reid* and *Walls* [5].

Let $a_\pm(b_\pm)$ be the annihilation operator for the horizontally $(+)$ or vertically $(-)$ polarised mode for the field travelling to analyser 1 (labelled 1) or analyser 2 (labelled 2). The state of the two photons may be written as

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(a_+^\dagger b_+^\dagger + a_-^\dagger b_-^\dagger)|0\rangle \; , \tag{13.3}$$

where $|0\rangle$ is the vacuum state. Using the notation $|n_1,\ n_2,\ n_3,\ n_4\rangle$ to denote n_1 photons in mode a_+, n_2 photons in mode a_-, n_3 photons in mode b_+ and n_4 photons in mode b_-, the state may be expressed as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1,0,1,0\rangle + |0,1,0,1\rangle) \, . \tag{13.4}$$

If the photon in analyser 1 is detected in the $(+)$ channel, the state of the photon directed towards 2 must be polarised in the horizontal direction. This correlation is thus precisely of the kind required for the EPR experiment.

We are free to measure the polarisation in any direction by rotating the analysers through the angles θ_1 and θ_2 for detector 1 and 2, respectively. The detected modes in this case are orthogonal transformations of the modes $a_\pm$ and $b_\pm$;

$$c_+ = a_+ \cos\theta_1 + a_- \sin\theta_1 \; , \tag{13.5a}$$

$$c_- = -a_+ \sin\theta_1 + a_- \cos\theta_1 \; , \tag{13.5b}$$

$$d_+ = b_+ \cos\theta_2 + b_- \sin\theta_2 \; , \tag{13.5c}$$

$$d_- = -b_+ \sin\theta_2 + b_- \sin\theta_2 \; . \tag{13.5d}$$

The detectors placed after the polarisers measure the intensities $\langle I_1^{\pm}\rangle$ and $\langle I_2^{\pm}\rangle$, while the correlators measure $\langle I_1^{+}I_2^{+}\rangle$, etc. In fact, for the two-photon state $\langle I_i^{\pm}\rangle = P_i^{\pm}$, is the probability for one count in the $+$ or $-$ channel of detector i. Of course, these moments depend on θ_1 and θ_2. Let us further suppose that in a complete theory these functions also depend on the variable λ which remains hidden from direct determination and for which only a statistical description is available. This variable is distributed according to some density $\rho(\lambda)$. In general, we may then write

$$\langle I_1^{\pm}I_2^{\pm}\rangle_{\theta_1\theta_2} = \int \rho(\lambda)I_1^{\pm}(\lambda,\theta_1,\theta_2)I_2^{\pm}(\lambda,\theta_1,\theta_2)\mathrm{d}\lambda \;, \tag{13.6}$$

where I_1^{+} denotes the expected intensity at detector 1 given a value for λ, namely

$$I_1^{+}(\lambda,\theta_1,\theta_2) = \int I_1^{+}\rho(I_1^{+}|\lambda,\theta_1,\theta_2)\mathrm{d}I_1^{+} \;. \tag{13.7}$$

It is reasonable to assume, as in EPR, that for a given value of λ the results at 1 cannot depend on the angle θ_2 chosen at 2, (and conversely). This is the “locality assumption”, it is formally represented by

$$I_1^{\pm}(\lambda,\theta_1,\theta_2) = I_1^{\pm}(\lambda,\theta_1) \;, \tag{13.8a}$$

$$I_2^{\pm}(\lambda,\theta_1,\theta_2) = I_2^{\pm}(\lambda,\theta_2) \;. \tag{13.8b}$$

Consider the following correlation functions:

$$E(\theta_1,\theta_2) = \frac{\langle (I_1^{+}-I_1^{-})(I_2^{+}-I_2^{-})\rangle}{\langle (I_1^{+}+I_1^{-})(I_2^{+}+I_2^{-})\rangle} \;. \tag{13.9}$$

In terms of the detected mode operators this may be written in the form

$$E(\theta_1,\theta_2) = \frac{\langle :(c_+^{\dagger}c_+ - c_-^{\dagger}c_-)(d_+^{\dagger}d_+ - d_-^{\dagger}d_-):\rangle}{\langle :(c_+^{\dagger}c_+ + c_-^{\dagger}c_-)(d_+^{\dagger}d_+ + d_-^{\dagger}d_-):\rangle} \tag{13.10}$$

where : : denotes normal ordering.

Assuming a local hidden variable theory we may write

$$E(\theta_1,\theta_2) = N^{-1}\int f(\lambda)S_1(\lambda,\theta_1)S_2(\lambda,\theta_2)\mathrm{d}\lambda \;, \tag{13.11}$$

where

$$S_1(\lambda,\theta_1) = \frac{I_1^{+}(\lambda,\theta_1) - I_1^{-}(\lambda,\theta_1)}{I_1(\lambda)} \;, \tag{13.12}$$

$$S_2(\lambda,\theta_1) = \frac{I_2^{+}(\lambda,\theta_2) - I_2^{-}(\lambda,\theta_2)}{I_2(\lambda)} \;, \tag{13.13}$$

$$f(\lambda) = \rho(\lambda) I_1(\lambda) I_2(\lambda) \tag{13.14}$$

with

$$I_1(\lambda) = I_1^+(\lambda, \theta_1) + I_1^-(\lambda, \theta_2)\,, \tag{13.15a}$$

$$I_2(\lambda) = I_2^+(\lambda, \theta_2) + I_2^-(\lambda, \theta_2)\,. \tag{13.15b}$$

The latter equations correspond to the intensity of light measured at 1 or 2 with the polarisers removed. The normalisation N is

$$N = \int f(\lambda)\mathrm{d}\lambda\,. \tag{13.16}$$

The functions $|S_1(\lambda, \theta_1)|$ and $|S_2(\lambda, \theta_2)|$ are bounded by unity:

$$|S_1(\lambda, \theta_1)| \leq 1\,, \tag{13.17a}$$

$$|S_2(\lambda, \theta_2)| \leq 1\,. \tag{13.17b}$$

To obtain a testable statistical quantity we need to consider how $E(\theta_1, \theta_2)$ changes as the orientation of the polarisers are changed. With this in mind, consider $E(\theta_1, \theta_2) - E(\theta_1, \theta_2')$. This quantity may be expressed as

$$\begin{aligned} E(\theta_1, \theta_2) - E(\theta_1, \theta_2') = N^{-1} &\int \mathrm{d}\lambda\, f(\lambda) S_1(\lambda, \theta_1) S_2(\lambda, \theta_2)[1 \pm S_1(\lambda, \theta_1') S_2(\lambda, \theta_2')] \\ &- N^{-1} \int \mathrm{d}\lambda\, f(\lambda) S_1(\lambda, \theta_1) S_2(\lambda, \theta_2') \\ &\times [1 \pm S_1(\lambda, \theta_1') S_2(\lambda, \theta_2)]\,. \end{aligned} \tag{13.18}$$

Then using (13.17a and b)

$$\begin{aligned} |E(\theta_1, \theta_2) - E(\theta_1, \theta_2')| \leq N^{-1} &\int \mathrm{d}\lambda\, f(\lambda)[1 \pm S_1(\lambda, \theta_1') S_2(\lambda, \theta_2')] \\ &+ N^{-1} \int \mathrm{d}\lambda\, f(\lambda)[1 \pm S_1(\lambda, \theta_1') S_2(\lambda, \theta_2)] \\ &= 2 \pm [E(\theta_1', \theta_2') + E(\theta_1', \theta_2)]. \end{aligned}$$

Finally, we obtain the Bell inequality

$$|B| \leq 2\,, \tag{13.19}$$

where

$$B = E(\theta_1, \theta_2) - E(\theta_1, \theta_2') + E(\theta_1', \theta_2') + E(\theta_1', \theta_2)\,.$$

This particular Bell inequality is known as the Clauser–Horne–Shimony–Holt (CHSH) inequality.

As we shall see, there are states of the field which violate the inequality equation (13.19) [for example, the state given in (5)]. We note firstly, however, that if the state of the field can be represented by a positive, normalisable Glauber–Sudarshan P-representation no violation of this inequality is possible. Let $\alpha = (\alpha_+, \alpha_-, \beta_+, \beta_-)$ be the c-number corresponding to the modes $a_\pm, b_\pm$. If we define the following 'transformation' variables for the modes $c_\pm, d_\pm$,

$$\begin{aligned} \gamma_+ &= \alpha_+ \cos\theta_1 + \alpha_- \sin\theta_1, & \delta_+ &= \beta_+ \cos\theta_2 + \beta_- \sin\theta_2, \\ \gamma_- &= -\alpha_+ \sin\theta_1 + \alpha_- \cos\theta_1, & \delta_- &= -\beta_+ \sin\theta_2 + \beta_- \cos\theta_2\,, \end{aligned} \tag{13.20}$$

the correlation function $E(\theta_1, \theta_2)$ becomes

$$E(\theta_1, \theta_2) = N^{-1} \int P(\alpha)(|\gamma_+|^2 - |\gamma_-|^2)(|\delta_+|^2 - |\delta_-|^2)\mathrm{d}^2\alpha \tag{13.21}$$

with

$$N = \int P(\alpha)(|\gamma_+|^2 + |\gamma_-|^2)(|\delta_+|^2 + |\delta_-|^2)\mathrm{d}^2\alpha\,.$$

Recalling that the transformations in (13.20) are orthogonal we note that

$$|\gamma_+|^2 + |\gamma_-|^2 = |\alpha_+|^2 + |\alpha_-|^2 \text{ and } |\delta_+|^2 + |\delta_-|^2 = |\beta_+|^2 + |\beta_-|^2$$

the normalisation may be written

$$N = \int P(\alpha)(|\alpha_+|^2 + |\alpha_-|^2)(|\beta_+|^2 + |\beta_-|^2)\mathrm{d}^2\alpha\,, \tag{13.22}$$

where the integrand does not depend on θ_1 or θ_2. Then

$$E(\theta_1, \theta_2) = N^{-1} \int P(\alpha)(|\alpha_+|^2 + |\alpha_-|^2)(|\beta_+|^2 + |\beta_-|^2)S(\gamma)S(\delta)\,, \tag{13.23}$$

where

$$S(\gamma) = \frac{|\gamma_+|^2 - |\gamma_-|^2}{|\alpha_+|^2 + |\alpha_-|^2} \tag{13.24}$$

and

$$S(\delta) = \frac{|\delta_+|^2 - |\delta_-|^2}{|\beta_+|^2 + |\beta_-|^2}\,. \tag{13.25}$$

As $S(\gamma)$ is a function of θ_1 and not θ_2 while $S(\delta)$ is a function of θ_2 and not θ_1, the Glauber–Sudarshan representation is local. It then follows immediately that provided $P(\alpha)$ is positive and normalisable, the Bell inequality in (13.19) must hold.

The correlation function $E(\theta_1, \theta_2)$ may be evaluated directly for the state in (13.4) using the normally-ordered moment in (13.9). One finds

$$E(\theta_1, \theta_2) = \cos 2\psi\,, \tag{13.26}$$

where

$$\psi \equiv \theta_1 - \theta_2 \,.$$

If we choose

$$\psi = \theta_2 - \theta_1 = \theta_1' - \theta_2 = \theta_1' - \theta_2' = \frac{1}{3}(\theta_1 - \theta_2') \,,$$

one finds

$$B = 3\cos 2\psi - \cos 6\psi \,. \tag{13.27}$$

When $\psi = 22.5°$, $B = 2\sqrt{2}$ showing a clear violation of the Bell inequality $|B| \leq 2$.

This violation has convincingly been demonstrated in the experiment of Aspect [4]. In this experiment the polarisation analysers were essentially beam splitters with polarisation-dependent transmittivity. Ideally, one would like to have the transmittivity (T^+) for the modes a_+ and b_+ equal to one, and the reflectivity (R^-) for the modes a_- and b_- also equal to one. However, in the experiment the measured values were $T_1^+ = R_1^- = 0.950$, $T_1^- = R_1^+ = 0.007$ and $T_2^+ = T_2^- = 0.930$, $T_2^- = R_2^+ = 0.007$.

The expression for $E(\theta_1, \theta_2)$ is then modified:

$$E(\theta_1, \theta_2) = F\frac{(T_1^+ - T_1^-)(T_2^+ - T_2^-)}{(T_1^+ + T_1^-)(T_2^+ + T_2^-)}\cos 2\psi \,, \tag{13.28}$$

where F is a geometrical factor accounting for finite solid angles of detection. In this experiment $F = 0.984$, and quantum mechanics would give for $\psi = 22.5°$, $B = 2.7$.

The observed value was 2.697 ± 0.015, in quite good agreement with quantum theory and a clear violation of the Bell inequality. In Fig. 13.2 is shown a plot of the theoretical and experimental results as a function of ψ. The agreement with quantum mechanics is better than 1%. It would appear in the light of this experiment that realistic local theories for completing quantum mechanics are untenable.

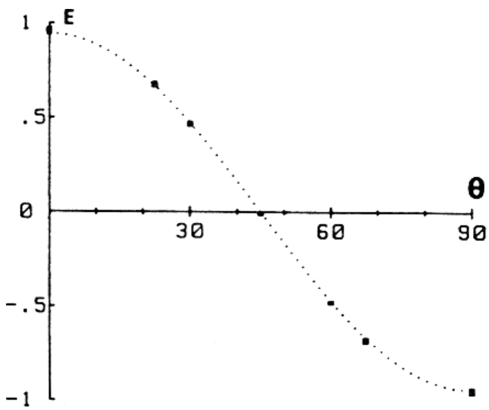


Fig. 13.2 Correlation of polarisations as a function of the relative angle of the polarisation analysers. The indicated errors are ± 2 standard deviations. The *dotted curve* is the quantum-mechanical prediction for the experiment. For ideal polarisers the curves woudl reach the values ± 1. (From Aspect et al. Phys. Rev. Letts. **49**, 92 (1982))

13.3 Violations of Bell's Inequalities Using a Parametric Amplifier Source

The Bell inequality presented in (13.19) is only one of a large class of inequalities violated by quantum mechanics. Another inequality has recently been tested by *Ou* and *Mandel* [6] based on an experiment first suggested by *Reid* and *Walls* [5]. It has now been realised in a number of configurations [7, 8]. We shall discuss the *Ou* and *Mandel* experiment presented schematically in Fig. 13.3. A parametric down converter produces two beams of linearly polarized signal and idler photons. Phase matching conditions give a relative angle of 4° between the propagation direction of the two beams. The idler photons pass through a 90° polarization rotator. The signal and idler beams are then incident from opposite sides onto a beam splitter. After the beam splitter, the two beams now consisting of mixed signal and idler photons pass through linear polarizers set at adjustable angles θ_1 and θ_2 before falling on two photodetectors. The coincidence counting rate of the two detectors is then measured with a time-to-digital converter. This provides a measure of the joint probability of detecting two photons for various settings θ_1 and θ_2 of the two polarizers.

In this experiment the polarisation analysers used have only a single output channel. However, we can still derive a Bell inequality violated by quantum mechanics. If we define the correlation function $P(\theta_1, \theta_2)$ by

$$P(\theta_1, \theta_2) = \langle I_1 I_2 \rangle_{\theta_1 \theta_2} \tag{13.29}$$

the following Bell inequality may be derived as

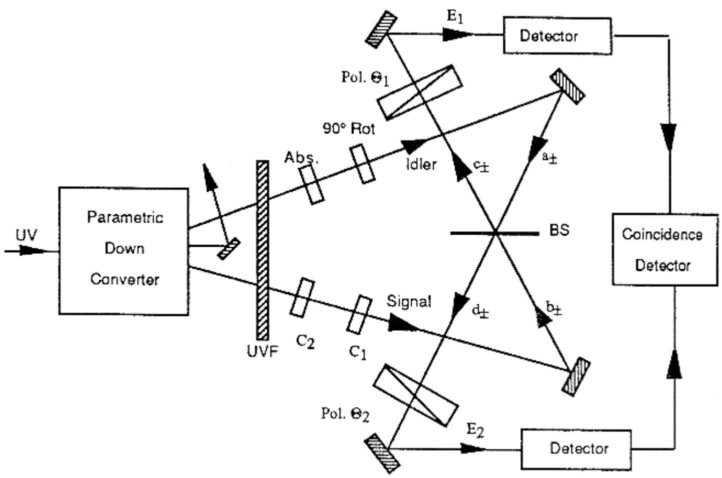


Fig. 13.3 Schematic representation of the experiment of *Ou* and *Mandel* to test the CHSH inequality using parametric down conversion [9]

$$S = P(\theta_1, \theta_2) - P(\theta_1, \theta_2') + P(\theta_1', \theta_2') + P(\theta_1', \theta_2)$$
$$- P(\theta_1, -) - P(-, \theta_2) \leq 0 , \tag{13.30}$$

where

$$P(\theta_1, -) = \langle I_1 I_2 \rangle_{\theta_1} , \tag{13.31}$$

$$P(-, \theta_2) = \langle I_1 I_2 \rangle_{\theta_2} \tag{13.32}$$

are the intensity correlation functions with one or the other polariser removed. The inequality in (13.30) is known as the Clauser–Horne inequality. Just as in the case of the CHSH inequality (13.19) this inequality is satisfied for states of the field which can be represented by a positive, normalisable Glauber–Sudarshan P-representation [5]. It may, however, be violated for certain quantum fields.

We follow closely the treatment given by *Tan* and *Walls* [9]. We now proceed to calculate $P(\theta_1, \theta_2)$ and S for the experiment of *Ou* and *Mandel* [6]. We include the possibility of placing an attenuator in the idler beam. Let a_+ and a_- denote the annihilation operators for the x and y polarized modes in the idler beam, and let b_+ and b_- denote the operators for the corresponding modes in the signal beam. The outgoing modes from the beam splitter are described by the operators $c_\pm$ and $d_\pm$, and obey the following relationships:

$$\begin{aligned} c_+ &= \sqrt{T_+} b_+ + \mathrm{i}\sqrt{R_+} a_+ , \\ c_- &= \sqrt{T_-} b_- - \mathrm{i}\sqrt{R_-} a_- , \\ d_+ &= \sqrt{T_+} a_+ + \mathrm{i}\sqrt{R_+} b_+ , \\ d_- &= \sqrt{T_-} a_- - \mathrm{i}\sqrt{R_-} b_- , \end{aligned} \tag{13.33}$$

where $T_\pm$ and $R_\pm$ are the intensity transmission and reflection coefficients for the x and y polarizations. The phase relationships arise from the Fresnel formula. Since the signal beam is polarized in the x direction, and the idler beam is polarized in the y direction, the modes associated with the operators a_+ and b_+ are the annihilation operators for the output modes of the parametric down converter. When an attenuator with the intensity transmission coefficient η is placed in the idler beam, the operators a_- in the above equations is replaced by

$$\sqrt{\eta} a_- + \sqrt{1-\eta}\, \upsilon , \tag{13.34}$$

where the vacuum mode operator υ is included to give the correct level of fluctuations in the attenuated beam. Photodetectors 1 and 2 respond to the fields $E_1^{(+)}$ and $E_2^{(+)}$, respectively, where

$$\begin{aligned} E_1^{(+)} &= c_+ \cos\theta_1 + c_- \sin\theta_1 , \\ E_2^{(+)} &= d_+ \cos\theta_2 + d_- \sin\theta_2 , \end{aligned} \tag{13.35}$$

The joint two-photon detection probability (for perfect detector efficiency) is

$$P(\theta_1,\theta_2) = \langle\psi|E_1^{(-)}E_2^{(-)}E_2^{(+)}E_1^{(+)}|\psi\rangle . \tag{13.36}$$

For low conversion efficiencies the output of the parametric down converter is a pair of photons, one in each of the signal and idler modes a_- and b_+. Thus $|\psi\rangle = |1,1\rangle$, this yields

$$P(\theta_1,\theta_2) = \eta(\sqrt{R_+R_-}\sin\theta_1\cos\theta_2 + \sqrt{T_+T_-}\cos\theta_1\sin\theta_2)^2 . \tag{13.37}$$

Taking a 50/50 beam splitter ($R_+ = R_- = T_+ = T_- = \frac{1}{2}$),

$$P(\theta_1,\theta_2) = \frac{1}{4}\eta\sin^2(\theta_1+\theta_2) . \tag{13.38}$$

Removing one polarizer, we must calculate

$$P(-,\theta_2) = \langle\psi| : (c_+^\dagger c_+ + c_-^\dagger c_-)E_2^{(-)}E_2^{(+)} : |\psi\rangle , \tag{13.39}$$

where : : represents normal ordering. For the input state $|1,1\rangle$ we find

$$P(-,\theta_2) = \frac{1}{4}\eta , \tag{13.40}$$

and similarly

$$P(\theta_1,-) = \frac{1}{4}\eta , \tag{13.41}$$

for a 50/50 beam splitter.

Substituting (13.38, 13.40 and 13.41) into the Clauser–Horne–Bell inequality (13.2) gives

$$S = \frac{1}{4}\eta[\sin^2(\theta_1+\theta_2) - \sin^2(\theta_1+\theta_2') + \sin^2(\theta_1'+\theta_2') + \sin^2(\theta_1'+\theta_2) - 2] .$$

Choosing the angles such that $\theta_1 = \pi/8, \theta_2 = \pi/4, \theta_1' = 3\pi/8$ and $\theta_2' = 0$,

$$S = \frac{1}{4}\eta(\sqrt{2}-1) > 0 , \tag{13.42}$$

which violates the inequality.

In a classical wave analysis of the parametric down converter, we represent the signal and idler fields incident on the beam splitter by the complex numbers E_s and E_i. The beam splitter combines these fields to produce E_1 and E_2 at the detectors, where

$$E_1 = \cos\theta_1 \sqrt{T_+} E_s - \mathrm{i} \sin\theta_1 \sqrt{R_-} E_i ,$$
$$E_2 = \mathrm{i} \cos\theta_2 \sqrt{R_+} E_s + \sin\theta_2 \sqrt{T_-} E_i \ . \tag{13.43}$$

The joint detection probability $P(\theta_1,\ \theta_2)$ is proportional to the intensity correlation $\langle |E_1|^2 E_2|^2\rangle$. Using the above forms for E_1 and E_2, and assuming that the difference in the phases of the signal and idler fields is random, we find that for a 50/50 beam splitter

$$P(\theta_1,\theta_2) \propto \langle I_s I_i\rangle \sin^2(\theta_1+\theta_2) + \langle I_s^2\rangle \cos^2\theta_1 \cos^2\theta_2 + \langle I_i^2\rangle \sin^2\theta_1 \sin^2\theta_2 \ . \tag{13.44}$$

where we have written I_s for $|E_s|^2$ and for I_i for $|E_i|^2$. With the attenuator in the idler beam, $I_i = \eta I_s$, and if we assume that the intensity fluctuations are such that $\langle I^2\rangle \propto \langle I\rangle^2$ and $\langle I_i I_s\rangle \propto \langle I_i\rangle\langle I_s\rangle$, then

$$P(\theta_1,\theta_2) \propto \eta \sin^2(\theta_1+\theta_2) + \cos^2\theta_1 \cos^2\theta_2 + \eta^2 \sin^2\theta_1 \sin^2\theta_2 \ . \tag{13.45}$$

In order to compare the quantum and classical result we consider $P(\theta,\ \pi/4)$ with $R_+ = R_- = T_+ = T_- = \frac{1}{2}$. Then

$$P(\theta,\pi/4) = \frac{\eta}{8}(1+\sin 2\theta) , \tag{13.46}$$

which exhibits a sinusoidal modulation with respect to the angle 2θ. The visibility of the resulting modulation is unity. However, the classical result gives

$$P(\theta,\pi/4) \propto \frac{\eta}{2}(1+\sin 2\theta) + \frac{1}{2}\cos^2\theta + \frac{\eta^2}{2}\sin^2\theta , \tag{13.47}$$

which in the absence of the absorber $(\eta = 1)$ gives

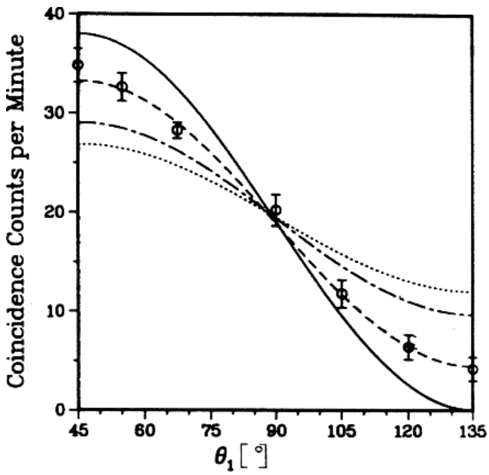
$$P(\theta,\pi/4) \propto (1+\frac{1}{2}\sin 2\theta) \ . \tag{13.48}$$

In the classical case the modulation is not 100%, in fact the visibility is only one half.

In the experiment of *Ou* and *Mandel* the value of S was found to be positive with an accuracy of six standard deviations, in clear violation of the Bell inequality (13.30). The experiment also distinguished between the different quantum and classical predictions for the phase dependence of $P(\theta,\pi/4)$. These results are shown in Fig. 13.4. The solid and dashed-dotted lines correspond to the quantum and classical wave predictions, respectively, with constants of proportionality adjusted to fit. Clearly, $P(\theta,\pi/4)$ does exhibit the phase dependence predicted by quantum mechanics. The observed visibility obtained from a best fit was 0.76; greater than the classical prediction of 0.5.

Instead of the correlated two-photon state discussed above we can also use the output state of a parametric down converter. We assume that the pump field in this

Fig. 13.4 The coincidence counting rate as a function of polariser angle θ_1 with θ_2 fixed at 45°. The *solid curve* is the quantum prediction based on (13.46), and the *dash-dot curve* is the classical prediction based on (13.48). The *dashed* and *dotted curves* are the quantum and classical predictions, respectively, including a detector inefficiency of 0.76



device may be treated classically. The solutions for the output modes of the device are

$$a_- = a_0 \cosh\kappa + b_0^\dagger \sinh\kappa \,, \tag{13.49}$$

$$b_+ = b_0 \cosh\kappa + a_0^\dagger \sinh\kappa \,, \tag{13.50}$$

where κ is proportional to the second-order nonlinear susceptibility of the crystal, and a_0, b_0 are the input modes. We assume that the input state is a vacuum. With a 50:50 beam splitter $\eta = 1$ and with $\theta_2' = 0, \theta_1 = \psi, \theta_2 = 2\psi, \theta_1' = 3\psi$ the quantity S which occurs in the Bell inequality (13.30) is given by (Exercise 13.2)

$$S = \frac{1}{4}\sinh^2\kappa\{F(\psi) + 2\sinh^2\kappa[F(\psi) + 2G(\psi)]\} \,, \tag{13.51}$$

where

$$F(\psi) = 2\sin^2 3\psi - \sin^2\psi + \sin^2 5\psi - 2,$$
$$G(\psi) = \sin^2\psi \sin^2 2\psi + \sin^2 3\psi \sin^2 2\psi - \sin^2 3\psi - \sin^2 2\psi \,.$$

When $\kappa \ll 1$ this may be approximated by

$$S \approx \frac{1}{4}\kappa^2 F(\psi) \,. \tag{13.52}$$

For purposes of comparison the two-photon state, with the same choice of angles would give

$$S = \frac{1}{4}F(\psi) \,. \tag{13.53}$$

Up to a scale constant in this limit the parametric down converter gives the same result as for a correlated photon pair.

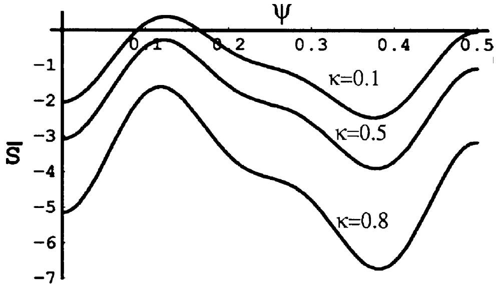


Fig. 13.5 The correlation function in (13.51) normalised by $\sinh^2\kappa$, versus ψ for various values of κ. The violation of the classical inequality is evident for small κ

In the limit $\kappa \gg 1$ we find

$$S \propto F(\psi) - 2G(\psi)\,. \tag{13.54}$$

As the function on the right-hand side is always nonpositive no violation of the Clauser–Horne inequality is possible. In Fig. 13.5 we plot S normalised by the intensity $I = \sinh^2 k$ versus ψ for various values of κ. We see that the maximum violation for $\kappa \ll 1$ occurs when $\theta = \pi/8$ (solid curve).

We note that the form of the intensity correlation function for the parametric down-converter in the limit of $\kappa \gg 1$ coincides with that of the classical analysis (13.44).

13.4 One-Photon Interference

In all the schemes discussed above the states which lead to a violation of the Bell inequalities are correlated two-photon states. We now consider a scheme which demonstrates the non-local nature of quantum mechanics, which does not rely on two-photon states. This experiment illustrates on the nonlocal behaviour of a single photon.

The scheme is illustrated in Fig. 13.6. A field is split at a 50:50 beam splitter, and each of the two output fields directed to homodyne detectors. Each of the homodyne detectors mix the output field from the first beam splitter with a coherent local oscillator of amplitude $\alpha_k = \alpha \mathrm{e}^{\mathrm{i}\theta_\kappa}$, and the final intensities at the two output channels of each of the homodyne detectors are measured using photodetectors. We follow closely the treatment of *Tan* et al. [10].

Referring to Fig. 13.6 we see that the homodyne detector k may be regarded as making a measurement of b_k with a local parameter θ_k. This parameter is analogous to the angle of the polarisation analysers in the two-photon schemes. We wish to determine the probabilities with which the individual photodetectors respond, and the coincidence probabilities for pairs of photodetectors, one in each homodyne detector.

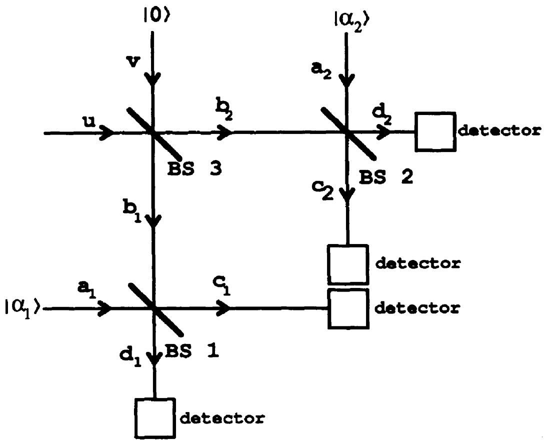


Fig. 13.6 Schematic representation of an experiment with a single photon state to demonstrate non-locality [10]

The transformation between the mode operators shown in Fig. 13.6 are given by

$$\begin{pmatrix} c_k \\ d_k \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix},$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix}. \tag{13.55}$$

Thus the modes input into the detectors may be expressed in terms of the input mode operators by

$$\begin{pmatrix} c_1 \\ d_1 \\ c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\mathrm{i}}{2} & 0 & -\frac{1}{2} \\ \frac{\mathrm{i}}{\sqrt{2}} & \frac{1}{2} & 0 & \frac{\mathrm{i}}{2} \\ 0 & -\frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{\mathrm{i}}{2} \\ 0 & \frac{\mathrm{i}}{2} & \frac{\mathrm{i}}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a_1 \\ v \\ a_2 \\ u \end{pmatrix}. \tag{13.56}$$

This enables us to calculate the coincidence probabilities between the detectors directly in terms of the input field.

We begin by considering vacuum inputs to the modes u and v. The local oscillators are assumed to be in coherent states $|\alpha e^{\mathrm{i}\theta_1}\rangle$, $|\alpha e^{\mathrm{i}\theta_2}\rangle$. The intensities at all detectors are found to be equal

$$\langle I_{c_1}\rangle = \langle I_{c_2}\rangle = \langle I_{d_1}\rangle = \langle I_{d_2}\rangle = \frac{1}{2}\alpha^2. \tag{13.57}$$

The two-photon coincidence rates due to rare chance coincidences between the local oscillators are also equal between the pairs of detectors

$$\langle I_{c_1} I_{c_2}\rangle = \langle I_{d_1} I_{d_2}\rangle = \langle I_{c_1} I_{d_2}\rangle = \langle I_{d_1} I_{c_2}\rangle = \frac{1}{4}\alpha^4. \tag{13.58}$$

We now consider the input of a single photon in mode u while the mode v is the vacuum. The state of the two-mode field (b_1 and b_2) after the first beam-splitter is then an entangled state of a one-photon state and the vacuum

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\mathrm{i}|1\rangle|0\rangle + |0\rangle|1\rangle) \tag{13.59}$$

which is precisely the same state as one gets (except for a phase factor) for a one-photon state incident on the two slits in Young's interference experiment.

The photon count probabilities at the individual detectors are now

$$\langle I_{c_1}\rangle = \langle I_{c_2}\rangle = \langle I_{d_1}\rangle = \langle I_{d_2}\rangle = \frac{1}{2}\alpha^2 + \frac{1}{4}\,. \tag{13.60}$$

Thus the intensities at each detector are increased by $\frac{1}{4}$, being the probability that the one-photon input is detected by any given detector. The coincidence count probabilities between the pairs of detectors differ, now depending on which is considered. We find

$$\langle I_{c_1}I_{c_2}\rangle = \langle I_{d_1}I_{d_2}\rangle = \frac{1}{4}\{\alpha^4 + \alpha^2[1 + \sin(\theta_1 - \theta_2)]\} \tag{13.61}$$

and

$$\langle I_{c_1}I_{d_2}\rangle = \langle I_{d_1}I_{c_2}\rangle = \frac{1}{4}\{\alpha^4 + \alpha^2[1 - \sin(\theta_1 - \theta_2)]\}\,. \tag{13.62}$$

The coincidence probabilities depend on the phase difference between the local oscillators $\theta_1 - \theta_2$. If this is set to $-\pi/2$, we get the minimum possible coincidence probability of $\frac{1}{4}\alpha^4$ between detector pairs $(c_1,\ c_2)$ and $(d_1,\ d_2)$ and the maximum coincidence probability of $\frac{1}{4}\alpha^4 + \frac{1}{2}\alpha^2$ between the pairs $(c_1,\ d_2)$ and $(d_1,\ d_2)$. We shall be most interested in the situation where α is small compared to one.

Let us first try to interpret these results from a naïve particle viewpoint. The great enhancement of the single count probability over that with vacuum inputs is easily understood by the above argument. On the other hand, a coincidence between two detectors is expected to be a rare event since there is only one incident photon, and a coincidence can only occur if an additional photon is generated by the (weak) local oscillator of the homodyne detector which the photon does *not* reach. Since these two photons are detected at two spatially separated detectors and have apparently arisen from independent sources, we would not expect any correlation between the paths of these photons within each homodyne detector. Nevertheless, the quantum mechanical analysis reveals that such a correlation is present. In fact, this correlation is so great that for the choice of phases given above, no additional coincidence (above the vacuum level) occur for particular detector pairs, whereas there is a relatively large coincidence probability (proportional to the local oscillator intensity) for the other pairs.

Non-local intensity correlation and their dependence on the local oscillator phases are not unexpected from a classical wave description of light. A classical analogue to the single photon input is a wave of low amplitude and unspecified phase. We may formally obtain the results for the classical wave theory from the quantum-mechanical calculation by substituting the wave amplitude $\beta e^{\pm i\phi}$ for b and $b^\dagger$, respectively, and averaging over the random phase ϕ. It is easy to check that the predicted average intensities and intensity correlations are given by

$$\langle I_{c_1}\rangle = \langle I_{c_2}\rangle = \langle I_{d_1}\rangle = \langle I_{d_2}\rangle = \frac{1}{2}\alpha^2 + \frac{1}{4}\beta^2 , \tag{13.63}$$

$$\langle I_{c_1}I_{c_2}\rangle = \langle I_{d_1}I_{d_2}\rangle = \frac{1}{4}\{\alpha^4 + \alpha^2\beta^2[1+\sin(\theta_1-\theta_2)] + \frac{1}{4}\beta^4\} , \tag{13.64}$$

$$\langle I_{c_1}I_{d_2}\rangle = \langle I_{d_1}I_{c_2}\rangle = \frac{1}{4}\{\alpha^4 + \alpha^2\beta^2[1-\sin(\theta_1-\theta_2)] + \frac{1}{4}\beta^4\} . \tag{13.65}$$

If we consider the coincidence probabilities as a function of $(\theta_1-\theta_2)$, we see that they can vary between $\frac{1}{4}(\alpha^4+\frac{1}{4}\beta^4)$ to $\frac{1}{4}(\alpha^4+2\alpha^2\beta^2+\frac{1}{4}\beta^4)$. This corresponds to a "visibility" of

$$v = \frac{\rho}{\rho^2+\rho+\frac{1}{4}} \tag{13.66}$$

where $\rho = (\alpha/\beta)^2$. The visibility attains a maximum value of $\frac{1}{2}$ when $\rho = \frac{1}{2}$. By contrast, the visibility as calculated from the quantum-mechanical result is

$$v = \frac{1}{\alpha^2+1} . \tag{13.67}$$

This can be made arbitrarily close to unity by choosing a sufficiently small value of α. Figure 13.7 shows the coincidence probabilities $\langle I_{c_1}I_{c_2}\rangle = \langle I_{d_1}I_{d_2}\rangle$ as a function of the local oscillator phase difference for the quantum mechanical and classical results with $\beta = 1$ and $\alpha = 1/\sqrt{2}$. This gives the same single count probability of $\frac{1}{2}$ in each detector, and the local oscillator amplitudes are optimized for maximum visibility

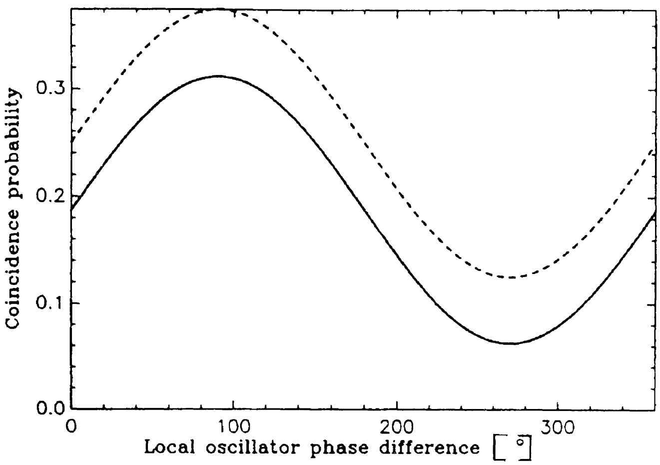


Fig. 13.7 Coincidence probability for the single photon non-locality experiment. The *solid line* is the quantum mechanical model, the *dashed line* is the prediction for a classical wave model [10]

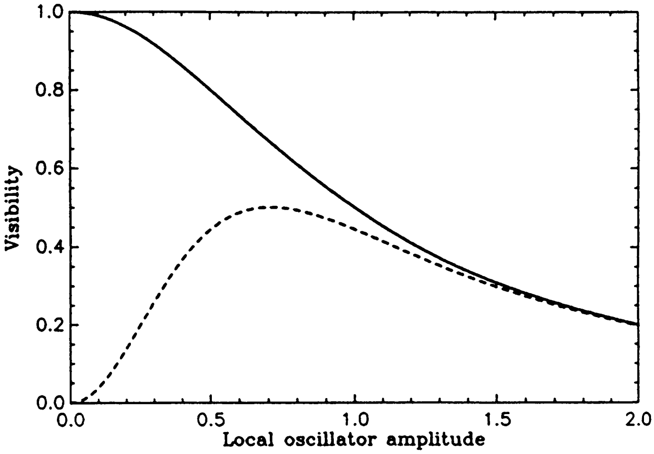


Fig. 13.8 Variation of visibility with the amplitude of the local oscillator for the quantum model (*solid line*) and a classical wave model (*dashed line*) [10]

in the classical result. However, the quantum mechanical visibility is considerably larger than that expected classically. This is clearly seen in Fig. 13.8 where the visibility v is plotted as a function of the coherent local oscillator amplitude α for the quantum mechanical single state and the classical wave mode with $\beta = 1$.

We thus see that by measuring the coincidence probability in a pair of detectors, it is possible to distinguish between the classical and quantum mechanical models. If the detector efficiencies are less than unity, coincidences will be missed, but this does not affect the measurement of the visibility of the effect.

Preparation of a single photon state may be achieved experimentally by using the signal beam of a parametric amplifier while monitoring photons in the idler beam [11]. *Hong* and *Mandel* [12] described an experiment in which a nearly pure single photon state was produced using this method. If the pump for the parametric amplifier is derived by frequency doubling a coherent beam, this provides a convenient source for the local oscillator required in the experiment under discussion.

In order to rigorously rule out classical explanations for the quantum mechanical result, it is necessary to show that Bell's inequality may be violated.

An intensity correlation coefficient is used which involves all four photo-detectors

$$E(\theta_1 - \theta_2) = \frac{\langle (I_{d_1} - I_{c_1})(I_{d_2} - I_{c_2})\rangle}{\langle (I_{d_1} + I_{c_1})(I_{d_2} + I_{c_2})\rangle} . \tag{13.68}$$

Evaluating this in terms of the statistics of the input mode u, where v is the vacuum yields

$$E(\theta_1 - \theta_2) = -\frac{\alpha^2\{\langle u^\dagger u\rangle \sin(\theta_2 - \theta_1) + |\langle u^2\rangle| \sin(\theta_2 + \theta_1 - \xi)\}}{\alpha^4 + \langle u^\dagger u\rangle\alpha^2 + \frac{1}{4}\langle u^{\dagger 2}u^2\rangle} \tag{13.69}$$

where $\langle u^2 \rangle = R\exp(\mathrm{i}\xi)$. When a single photon input is considered for u, this reduces to

$$E(\theta_1, \theta_2) = \frac{1}{\alpha^2+1}\sin(\theta_1 - \theta_2)\,. \tag{13.70}$$

If th coefficient of $\sin(\theta_1 - \theta_2)$ is greater than $1/\sqrt{2}$ it is well-known that this functional form for the correlation allows a violation of Bell's inequalities. This is clearly possible if α is made sufficiently small. It has been shown [13] that such a violation of Bell's inequalities is not possible if u is in a coherent state, no matter how small the input amplitude may be.

In conclusion, some of the most striking features of non-locality in quantum mechanics may be demonstrated using phase-sensitive measurements on the field produced by a single photon. These effects may not be explained classically using a particle, wave or hidden-variable theory involving local causality.

Exercises

13.1 Derive (13.26) for the correlation function $E(\theta_1,\ \theta_2)$. Show that with the choice $\psi = \theta_2 - \theta_1 = \theta_1' - \theta_2 = \theta_1' - \theta_2' = \frac{1}{3}(\theta_1 - \theta_2')$ one obtains (13.27) for **B**.

13.2 Derive (13.51) for the Bell parameter S for the parametric amplifier .

13.3 The state going from the beam splitter in the one-photon interference experiment is the linear superposition state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\mathrm{i}|1\rangle|0\rangle + |0\rangle|1\rangle)\,.$$

Compute the intensity correlations were this state replaced by the mixed state

$$\rho = \frac{1}{2}(|1\rangle_u\langle 1| \otimes |0\rangle_v\langle 0| + |0\rangle_u\langle 0| \otimes |1\rangle_v\langle 1|)$$

and show that no violation of the Bell inequality can occur.

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